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Exact solution of the random bipartite matching model

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Abstract. In this paper we present the exact solution for the average minimum energy of the random bipartite matching model with an arbitrary finite number of elements where randomly paired interactions are described by an independent exponential distribution. This solution confirms the Parisi conjecture proposed for this model previously, as well as the result of the replica solution of this model in the thermodynamic limit.

1. The model

The model under consideration can be formulated as follows. We have a society consisting of N 'men' (labelled by $i = 1, 2, \ldots, N$) and N 'women' (labelled by $j = 1, 2, \ldots, N$) described by a given set of N^2 random non-negative interactions $\{J_{ij}\}$ between every man and every woman. The statistics of J_{ij}s is defined by a probability distribution function $P[J_{ij}]$.

Then we consider all possible 'marriages' with strict monogamy: every man can be connected with one and only one woman, and vice versa. Thus, a particular marriage configuration in this society can be described by the $N \times N$ permutation matrix S_{ij} with the elements taking values 0 or 1 ('0' for all non-coupled pairs of men and women, and '1' for married couples) constrained by two conditions:

$$\sum_{i=1}^{N} S_{ij} = \sum_{j=1}^{N} S_{ij} = 1 \tag{1.1}$$

which allow one and only one '1' in each row and in each column of the matrix $\hat{S}$. The total number of all possible marriage configurations in this society is thus equal to $N!$.

Now for every marriage configuration $\hat{S}$ we introduce the total energy, or total weight (the Hamiltonian):

$$H[\hat{S}; \hat{J}] = \sum_{i,j=1}^{N} S_{ij} J_{ij}. \tag{1.2}$$

For a given matrix $\hat{S}$ this energy is equal to the sum of N particular J_{ij}s (one from each line and each column) corresponding to the particular married couples. In this paper we consider

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the simplest possible model in which the interactions $\{J_{ij}\}$ are assumed to be independent and described by the bounded exponential distribution

$$P[J_{ij}] = \prod_{i,j=1}^{N} \exp(-J_{ij}) \qquad (0 \leqslant J_{ij} < +\infty). \tag{1.3}$$

The problem studied below is formulated as follows: one has to find the value E_N of the average (over the distribution $P[J_{ij}]$) *minimum* (over all configurations of the permutation matrix S_{ij}) energy (1.2):

$$E_N = \left[\prod_{i,j=1}^{N} \int_0^\infty \mathrm{d}J_{ij}\right] P[J_{ij}] \min_{S_{ij}} \left(\sum_{i,j=1}^{N} S_{ij} J_{ij}\right). \tag{1.4}$$

Equivalently, in the language of statistical mechanics E_N can be obtained as the zero-temperature limit of the average free energy:

$$\begin{aligned} E_N &= -\lim_{\beta\to\infty} \frac{1}{\beta} \left[\prod_{i,j=1}^{N} \int_0^\infty \mathrm{d}J_{ij} \exp(-J_{ij})\right] \log\left(\sum_{S_{ij}} \exp\left[-\beta \sum_{i,j=1}^{N} S_{ij} J_{ij}\right]\right) \\ &\equiv -\lim_{\beta\to\infty} \frac{1}{\beta} \overline{\left(\log\left[\sum_{S_{ij}} \exp\{-\beta H[\hat{S}; \hat{J}]\}\right]\right)}. \end{aligned} \tag{1.5}$$

Thus, we face the typical problem of statistical mechanics with quenched disorder: first, for given values of random parameters $\{J_{ij}\}$ one has to compute the partition function and the free energy, and only after that does one carry out the averaging over J_{ij}s.

In the thermodynamic limit ($N \to \infty$) this problem has been solved some years ago in the framework of the replica symmetric ansatz [1], yielding the result

$$E_{N\to\infty} = \zeta(2) = \tfrac{1}{6}\pi^2. \tag{1.6}$$

In this paper we present the exact solution of this problem for an arbitrary (finite) value of N.

The case $N = 1$ is trivial,

$$E_{N=1} = 1. \tag{1.7}$$

The case $N = 2$ is only slightly more complicated, and it can also be easily calculated explicitly. Here the 2×2 permutation matrix $\hat{S}$ can have only two configurations:

$$\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \tag{1.8}$$

and

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array}. \tag{1.9}$$

Thus, according to the definitions (1.4) or (1.5) we have

$$\begin{aligned} E_{N=2} = 2 \int_0^\infty & \mathrm{d}J_{11}\,\mathrm{d}J_{12}\,\mathrm{d}J_{21}\,\mathrm{d}J_{22}\,(J_{11} + J_{22}) \exp\{-J_{11} - J_{12} - J_{21} - J_{22}\} \\ & \times\theta(J_{12} + J_{21} - J_{11} - J_{22}). \end{aligned} \tag{1.10}$$

Here the θ-function ensures that the state (1.8) has a lower energy than (1.9) (due to the obvious symmetry of the system the contribution from the opposite situation turns out to be the same, and this provides the factor of two in the above equation). Simple integration yields

$$E_{N=2} = 1 + \tfrac{1}{4}. \tag{1.11}$$

Noting that the result (1.6) for $N = \infty$ can also be represented in the form

$$E_{N\to\infty} = \zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} \tag{1.12}$$

and taking into account the results (1.7) and (1.11), Parisi has recently proposed a very elegant conjecture that the solution of the problem for an arbitrary value of N must be the following [2]:

$$E_N = \sum_{k=1}^{N} \frac{1}{k^2}. \tag{1.13}$$

For the *direct* calculation of E_N (in the style of equation (1.10)) with an arbitrary N one should perform the integration over the parameters $\{J_{ij}\}$ in the constrained positive subspace $J_{ij} \geqslant 0$ of the N^2-dimensional space. Since the total number of states of the $N \times N$ permutation matrix is equal to $N!$ this integration is also constrained by $(N! - 1)$ hyperplanes which guarantee that one chosen particular state has the minimum energy. One can easily verify that even in the case $N = 3$ such a calculation turns out to be an extremely difficult problem. Nevertheless, simple numerical tests for $N = 3, 4, 5$ proved to be compatible with the above conjecture with the precision $\sim 10^{-5}$ [2]. Moreover, recent analytical studies have provided the exact solution of this problem for $N \leqslant 4$ [3] and for $N \leqslant 6$ [4], and the result of these solutions confirms the conjecture (1.13). Here we use the original idea (proposed by Bravyi) of the unpublished work [3] to prove that the conjecture (1.13) is indeed correct for arbitrary N.

2. The proof

To ease further presentation of the proof let us introduce the following notation. The operation of finding the average of the minimum energy of the $N \times N$ problem (defined in equations (1.4) or (1.5)) will be denoted by the symbol

$$\boldsymbol{E}\left(\begin{array}{|c|c|c|c|c|c|}\hline \; & \; & \; & \; & \cdots & \; \\ \hline & & & & \cdots & \\ \hline & & & & \cdots & \\ \hline & & & & \cdots & \\ \hline \multicolumn{6}{|c|}{\cdots} \\ \hline & & & & \cdots & \\ \hline \end{array}\right) = E_N. \tag{2.1}$$

It is assumed that 'empty' boxes in the above matrix actually contain random elements $\{J_{ij}\}$

Let us consider the first line of the random matrix J_{ij}, and among N of its elements J_{1j} let us find the minimum one: $J^{(1)} \equiv \min_j(J_{1j})$. Due to its obvious symmetry of the problem with respect to permutations of the columns of the matrix J_{ij} we can always place this minimum element in the position (1, 1). Now let us redefine the elements of the *first* line as follows:

$$J_{1j} = J^{(1)} + \tilde{J}_{1j} \qquad (j \neq 1) \tag{2.2}$$

and leave all the other elements unchanged. According to (1.3), the elements $\tilde{J}_{1j}$ are described by *the same* exponential distribution, $P[\tilde{J}_{1j}] = \exp(-\tilde{J}_{1j})$, $(\tilde{J}_{1j} \geqslant 0)$, while for $J^{(1)}$ the distribution is

$$P[J^{(1)}] = N \exp\left(-N J^{(1)}\right). \tag{2.3}$$

Due to the constrains (1.1), the above redefinition produces only a simple shift of the Hamiltonian (1.2):

$$H = J^{(1)} + \sum_{i,j=1}^{N} S_{ij} \tilde{J}_{ij} \tag{2.4}$$

where the random matrix $\tilde{J}_{ij}$ contains '0' in the position $(1, 1)$, while the rest of its elements are described by the same distribution (1.3). Now using the definition of E_N, equation (1.5), we can easily integrate out $J^{(1)}$ to obtain

$$E_N = \frac{1}{N} + E_N^{(1)} \tag{2.5}$$

where

$$E_N^{(1)} = \boldsymbol{E} \left(\begin{array}{|c|c|c|c|c|c|} \hline 0 & & & & \cdots & \\ \hline & & & & \cdots & \\ \hline & & & & \cdots & \\ \hline & & & & \cdots & \\ \hline \multicolumn{6}{|c|}{\cdots} \\ \hline & & & & \cdots & \\ \hline \end{array} \right). \tag{2.6}$$

To calculate $E_N^{(1)}$ let us consider the *second* line of the above random matrix, and among N of its elements J_{2j} let us find the minimum one: $J^{(2)} \equiv \min_j(J_{2j})$. Now, due to the '0' in position $(1, 1)$, the first column of this matrix is no longer equivalent to the rest of the $(N-1)$ columns (which remain equivalent among themselves). Therefore, with probability $1/N$ the minimum element can be in the position $(2, 1)$, and with the probability $(N-1)/N$ it can be in the rest of the positions of the second line, and in this last case we can place it in position $(2, 2)$. Then we shift the values of the elements of the second line: $J_{2j} = J^{(2)} + \tilde{J}_{2j}$ (which leave the distribution of $\{\tilde{J}_{2j}\}$ unchanged). The integration over $J^{(2)}$ gives one more factor $1/N$, and for E_N we obtain

$$E_N = \frac{2}{N} + \frac{(N-1)}{N} E_N^{(2)} + \frac{1}{N} \tilde{E}_N^{(2)} \tag{2.7}$$

where

$$E_N^{(2)} = \boldsymbol{E} \left(\begin{array}{|c|c|c|c|c|c|} \hline 0 & & & & \cdots & \\ \hline & 0 & & & \cdots & \\ \hline & & & & \cdots & \\ \hline & & & & \cdots & \\ \hline \multicolumn{6}{|c|}{\cdots} \\ \hline & & & & \cdots & \\ \hline \end{array} \right) \tag{2.8}$$

and

$$\tilde{E}_N^{(2)} = \boldsymbol{E} \left(\begin{array}{|c|c|c|c|c|c|} \hline 0 & & & & \cdots & \\ \hline 0 & & & & \cdots & \\ \hline & & & & \cdots & \\ \hline & & & & \cdots & \\ \hline \multicolumn{6}{|c|}{\cdots} \\ \hline & & & & \cdots & \\ \hline \end{array} \right). \tag{2.9}$$

Equation (2.7) can be represented in the form

$$E_N = \frac{2}{N} + E_N^{(2)} + \frac{1}{N}\delta E_N^{(2)} \tag{2.10}$$

where

$$\delta E_N^{(2)} = \boldsymbol{E}\left(\begin{array}{|c|c|c|c|c|c|}\hline 0 & & & & \cdots & \\ \hline 0 & & & & \cdots & \\ \hline & & & & \cdots & \\ \hline & & & & \cdots & \\ \hline \multicolumn{6}{|c|}{\cdots} \\ \hline & & & & \cdots & \\ \hline \end{array}\right) - \boldsymbol{E}\left(\begin{array}{|c|c|c|c|c|c|}\hline 0 & & & & \cdots & \\ \hline & 0 & & & \cdots & \\ \hline & & & & \cdots & \\ \hline & & & & \cdots & \\ \hline \multicolumn{6}{|c|}{\cdots} \\ \hline & & & & \cdots & \\ \hline \end{array}\right). \tag{2.11}$$

To calculate the value $E_N^{(2)}$ defined by the matrix

$$\begin{array}{|c|c|c|c|c|c|}\hline 0 & & & & \cdots & \\ \hline & 0 & & & \cdots & \\ \hline & & & & \cdots & \\ \hline & & & & \cdots & \\ \hline \multicolumn{6}{|c|}{\cdots} \\ \hline & & & & \cdots & \\ \hline \end{array} \tag{2.12}$$

let us consider its *third* line, and among N elements J_{3j} let us find the minimum one: $J^{(3)} \equiv \min_j(J_{3j})$. Due to the two '0's in positions $(1, 1)$ and $(2, 2)$, the first and the second columns of this matrix are equivalent between themselves, but they are not equivalent to the rest of the $(N-2)$ columns (which remain equivalent among themselves). Therefore, with the probability $2/N$ the minimum element can be placed in the position $(3, 2)$, and with probability $(N-2)/N$ it can be in the rest of the positions of the third line, and here we can place it in position $(3, 3)$. Then we shift the values of the elements of the third line: $J_{3j} = J^{(3)} + \tilde{J}_{3j}$ (which again leave the distribution of $\{\tilde{J}_{3j}\}$ unchanged), and integrate over $J^{(3)}$ which gives one more factor $1/N$. In this way we obtain

$$E_N = \frac{3}{N} + \frac{(N-2)}{N}E_N^{(3)} + \frac{2}{N}\tilde{E}_N^{(3)} + \delta E_N^{(2)} \tag{2.13}$$

where

$$E_N^{(3)} = \boldsymbol{E}\left(\begin{array}{|c|c|c|c|c|c|}\hline 0 & & & & \cdots & \\ \hline & 0 & & & \cdots & \\ \hline & & 0 & & \cdots & \\ \hline & & & & \cdots & \\ \hline \multicolumn{6}{|c|}{\cdots} \\ \hline & & & & \cdots & \\ \hline \end{array}\right) \tag{2.14}$$

and

$$\tilde{E}_N^{(3)} = \boldsymbol{E}\left(\begin{array}{|c|c|c|c|c|c|}\hline 0 & & & & \cdots & \\ \hline & 0 & & & \cdots & \\ \hline & 0 & & & \cdots & \\ \hline & & & & \cdots & \\ \hline \multicolumn{6}{|c|}{\cdots} \\ \hline & & & & \cdots & \\ \hline \end{array}\right). \tag{2.15}$$

Equation (2.13) can be represented in the form

$$E_N = \frac{3}{N} + E_N^{(3)} + \frac{2}{N}\delta E_N^{(3)} + \frac{1}{N}\delta E_N^{(2)} \tag{2.16}$$

where

$$\delta E_N^{(3)} = \boldsymbol{E}\left(\begin{array}{|c|c|c|c|c|c|}\hline 0 & & & & \cdots & \\ \hline & 0 & & & \cdots & \\ \hline & 0 & & & \cdots & \\ \hline & & & & \cdots & \\ \hline \multicolumn{6}{|c|}{\cdots} \\ \hline & & & & \cdots & \\ \hline \end{array}\right) - \boldsymbol{E}\left(\begin{array}{|c|c|c|c|c|c|}\hline 0 & & & & \cdots & \\ \hline & 0 & & & \cdots & \\ \hline & & 0 & & \cdots & \\ \hline & & & & \cdots & \\ \hline \multicolumn{6}{|c|}{\cdots} \\ \hline & & & & \cdots & \\ \hline \end{array}\right). \tag{2.17}$$

Proceeding in this way up to the last line we eventually obtain

$$E_N = 1 + \sum_{k=2}^{N} \frac{k-1}{N}\delta E_N^{(k)} \tag{2.18}$$

(note that $E_N^{(N)} \equiv 0$ since it is given by the matrix with all zeros on the diagonal) where

$$\delta E_N^{(k)} = \boldsymbol{E}\left(\begin{array}{|c|c|c|c|c|c||c|c|c|}\hline 0 & & \cdots & & & & & \cdots & \\ \hline & 0 & \cdots & & & & & \cdots & \\ \hline \multicolumn{9}{|c|}{\cdots} \\ \hline & & \cdots & 0 & & & & \cdots & \\ \hline & & \cdots & & 0 & & & \cdots & \\ \hline & & \cdots & & 0 & & & \cdots & \\ \hline\hline & & \cdots & & & & & \cdots & \\ \hline \multicolumn{9}{|c|}{\cdots} \\ \hline & & \cdots & & & & & \cdots & \\ \hline \end{array}\right)$$

$$-\boldsymbol{E}\left(\begin{array}{|c|c|c|c|c|c||c|c|c|}\hline 0 & & \cdots & & & & & \cdots & \\ \hline & 0 & \cdots & & & & & \cdots & \\ \hline \multicolumn{9}{|c|}{\cdots} \\ \hline & & \cdots & 0 & & & & \cdots & \\ \hline & & \cdots & & 0 & & & \cdots & \\ \hline & & \cdots & & & 0 & & \cdots & \\ \hline\hline & & \cdots & & & & & \cdots & \\ \hline \multicolumn{9}{|c|}{\cdots} \\ \hline & & \cdots & & & & & \cdots & \\ \hline \end{array}\right). \tag{2.19}$$

Here the double lines mark the positions of the kth column and the kth line.

It can be proved (see appendix A) that the above value $\delta E_N^{(k)}$ is given by the *rectangular* $N \times k$ random matrix problem:

$$\delta E_N^{(k)} = \boldsymbol{E}\left(\begin{array}{|c|c|c|c|c|c||c|c|c|}\hline 0 & & \cdots & & & & & \cdots & \\ \hline & 0 & \cdots & & & & & \cdots & \\ \hline \multicolumn{9}{|c|}{\cdots} \\ \hline & & \cdots & 0 & & & & \cdots & \\ \hline & & \cdots & & 0 & & & \cdots & \\ \hline & & \cdots & & 0 & & & \cdots & \\ \hline\hline \end{array}\right) \tag{2.20}$$

defined by the Hamiltonian

$$H[\hat{S}; \hat{J}] = \sum_{i=1}^{N} \sum_{j=1}^{k} S_{ij} J_{ij} \tag{2.21}$$

where the random matrix J_{ij} is shown in equation (2.20) (with the same independent exponential distributions of non-zero elements). Here the 'truncated' $N \times k$ part of the original permutation matrix $\hat{S}$ again can have only one '1' in each line, and besides it has k columns each containing only one '1' and $(N - k)$ columns each containing only '0'.

It turns out that the above 'rectangular' problem, equation (2.20), can be solved explicitly (the proof is given in appendix B):

$$\delta E_N^{(k)} = \frac{1}{k(k-1)} \sum_{l=1}^{k-1} \frac{l}{N-l}. \tag{2.22}$$

Substituting this result into equation (2.18) we find

$$E_N = 1 + \frac{1}{N} \sum_{k=2}^{N} \frac{1}{k} \sum_{l=1}^{k-1} \frac{l}{N-l}. \tag{2.23}$$

After some simple algebra one eventually finds

$$E_N - E_{N-1} = \frac{1}{N^2} \tag{2.24}$$

which proves the result (1.13).

It should be noted in conclusion that the obtained solution is only valid for the considered exponential-type distribution, equation (1.3). It is crucial for the above proof that the form of the distribution of a random element J_{ij} does not change after its shift by a constant value. On the other hand, it is clear from the above proof that in the thermodynamic limit $N \to \infty$ the leading (in $1/N$) contribution to E_N is defined only by the very beginning of the distribution, $P[J \to 0]$. Therefore, the result $E_{N\to\infty} = \zeta(2)$ must also be correct for the 'rectangular'-type distribution: $P[0 \leqslant J \leqslant 1] = 1$; $P[J > 1] = 0$ (it is actually the model with this type of distribution which was studied in the replica solution [1]). For the discussion of other types of matching models see, e.g., [5] and references therein.

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Appendix A

In this appendix we prove that the value of $\delta E_N^{(k)}$ defined in equation (2.19) is given by the rectangular $N \times k$ problem (2.20).

First, let us consider the simplest case $k = 2$:

$$\delta E_N^{(2)} = \boldsymbol{E}\left(\begin{array}{|c||ccccccc|} \hline 0 & & & & & & \cdots & \\ \hline 0 & & & & & & \cdots & \\ \hline\hline & & & & & & \cdots & \\ \hline & & & & & & \cdots & \\ \hline & & & & & & \cdots & \\ \hline & & & & & & \cdots & \\ \hline \multicolumn{8}{|c|}{\cdots} \\ \hline & & & & & & \cdots & \\ \hline \end{array}\right) - \boldsymbol{E}\left(\begin{array}{|cc||cccccc|} \hline 0 & & & & & & \cdots & \\ \hline & 0 & & & & & \cdots & \\ \hline\hline & & & & & & \cdots & \\ \hline & & & & & & \cdots & \\ \hline & & & & & & \cdots & \\ \hline & & & & & & \cdots & \\ \hline \multicolumn{8}{|c|}{\cdots} \\ \hline & & & & & & \cdots & \\ \hline \end{array}\right)$$

$$\equiv \tilde{E}_N^{(2)} - E_N^{(2)}. \tag{A.1}$$

The above two problems, $\tilde{E}_N^{(2)}$ and $E_N^{(2)}$, differ only by the permutation of two elements: $(2, 1)$ and $(2, 2)$, while all the other matrix elements J_{ij} in both matrices are the same. Nevertheless, even this 'tiny' permutation, in general, can make the ground state configurations of the matrix $\hat{S}$ in the two problems are quite different. Note that for the calculation of the above average energy difference $\delta E_N^{(2)}$ we can average over J_{ij} both simultaneously (keeping J_{ij} to be the same in both problems) as well as separately for $\tilde{E}_N^{(2)}$ and for $E_N^{(2)}$.

For further proof it is important to introduce the concept of *equivalence* among the columns (and among the lines). We call the two columns j_1 and j_2 (or the two lines i_1 and i_2) equivalent if the probabilities of the positions (i, j_1) and (i, j_2) (or (i_1, j) and (i_2, j)) to be occupied in the ground state are equal.

Due to the obvious symmetry properties of the systems under consideration, it is evident that in each of the above problems, $\tilde{E}_N^{(2)}$ and $E_N^{(2)}$, all the columns on the right of the double vertical line, and all the lines below the double horizontal line are equivalent among themselves. On the other hand, the first two lines in each of the above problems are also equivalent between themselves, but they are not equivalent to the rest of the $(N-2)$ lines. Besides, in the problem $E_N^{(2)}$ we have the first two columns which are equivalent between themselves, but which are not equivalent to the rest of the $(N-2)$ columns. Finally, in the problem $\tilde{E}_N^{(2)}$ the first column is not equivalent to the rest of the $(N-1)$ columns.

Now we can separate all possible ground state configurations of the two problems, $\tilde{E}_N^{(2)}$ and $E_N^{(2)}$, into several *non-equivalent* classes according to the positions of the occupied elements in the first two lines.

Due to the equivalence of the first two lines and due to the equivalence of the $(N-2)$ columns $(j = 3, \ldots, N)$ we can reduce all the ground states of the problem $E_N^{(2)}$ to the following four non-equivalent basic configurations:

$$\begin{array}{cc} \begin{array}{|cc||ccc} \hline 0 & & & \bullet & \\ \hline & 0 & \bullet & & \\ \hline\hline & & & & \end{array} & \begin{array}{|cc||ccc} \hline 0 & \bullet & & & \\ \hline & 0 & \bullet & & \\ \hline\hline & & & & \end{array} \\ (a) & (b) \\ \begin{array}{|cc||ccc} \hline \odot & & & & \\ \hline & 0 & \bullet & & \\ \hline\hline & & & & \end{array} & \begin{array}{|cc||ccc} \hline \odot & & & & \\ \hline & \odot & & & \\ \hline\hline & & & & \end{array} \\ (c) & (d) \end{array} \tag{A.2}$$

where '$\bullet$' represent the elements occupied in the ground state configuration of the matrix $\hat{S}$, and '$\odot$' denote occupied element with '0'. Note that each of the above configurations represents the whole set of equivalent configurations. For instance, (A.2*b*) represents all configurations with '•' in any of $(N-2)$ positions $(2, j)$, $(j = 3, \ldots, N)$, as well as all configurations with '•' in the position $(2, 1)$ and another '•' in any of $(N-2)$ positions $(1, j)$, $(j = 3, \ldots, N)$. The diagram (A.2*c*) represents all configurations in which *one* of the zeros is occupied. Note also that all configurations of the type

$$\begin{array}{|c|c||c|c|c} 0 & \bullet & & & \\ \hline \bullet & 0 & & & \\ \hline\hline & & & & \end{array} \qquad \text{(A.3)}$$

must be excluded from the consideration since they *cannot* be the ground state as they are always higher in energy than the states represented in (A.2*d*).

In the same way, due to the equivalence of the first two lines and due to the equivalence of $(N-1)$ columns $(j = 2, \ldots, N)$ in the problem $\tilde{E}_N^{(2)}$ we have only two non-equivalent basic configurations:

$$\underset{(a)}{\begin{array}{|c||c|c|c|c} 0 & * & & & \\ \hline 0 & & * & & \\ \hline\hline & & & & \end{array}} \qquad \underset{(b)}{\begin{array}{|c||c|c|c|c} \odot & & & & \\ \hline 0 & * & & & \\ \hline\hline & & & & \end{array}}. \qquad \text{(A.4)}$$

Here for the occupied positions we use the notation '$*$' instead of '•' to distinguish them from those in the ground states of the problem $E_N^{(2)}$.

Now to compute the contribution to the difference of the energies $\delta E_N^{(2)}$, equation (A.1), we have to consider all possible combinations of the ground state configurations of the problem $E_N^{(2)}$, equation (A.2), and of those of the problem $\tilde{E}_N^{(2)}$, equation (A.4).

It is evident that if in the problem $E_N^{(2)}$ we have one of the configurations of the type (A.2*a*) or (A.2*b*) and in the problem $\tilde{E}_N^{(2)}$ we have one of the configurations of the type (A.4*a*) (all those in which no one '0' is occupied), then (since the two problems contain the same set of J_{ij}s) the positions of '•' and '$*$' in the first two lines (as well as all occupied positions in the rest of $(N-2)$ lines) must coincide. Therefore, these two cases give no contribution to $\delta E_N^{(2)}$, equation (A.1).

It is also evident that the combination of one of the ground states of the type (A.2*a*) or (A.2*b*) with (A.4*b*) is impossible. For example, let us suppose that the ground state of the problem $E_N^{(2)}$ is the configuration (A.2*a*), and that of the problem $\tilde{E}_N^{(2)}$ is the configuration (A.4*b*). Then, according to the definition of the ground state, the energy of (A.2*a*) must be smaller than that of the configuration (A.2*d*), which in turn (since the problem $\tilde{E}_N^{(2)}$ contain the same set of J_{ij}s) must be smaller than the energy of the configuration (A.4*b*). On the other hand, the energy of the configuration (A.2*a*) is equal to

$$\begin{array}{|c||c|c|c|c} 0 & & & * & \\ \hline 0 & & * & & \\ \hline\hline & & & & \end{array} \qquad \text{(A.5)}$$

of the problem $\tilde{E}_N^{(2)}$. Thus, the energy of (A.5) is smaller than that of (A.4*b*), and therefore (A.4*b*) cannot be the ground state.

Similar arguments show that the combinations of (A.2*c*) with (A.4*a*), as well as (A.2*d*) with (A.4*a*) are also impossible.

The combination of (A.2*c*) and (A.4*b*) is allowed, but in this case, according to the definition of the ground state, the position of '•' in (A.2*c*) of the problem $E_N^{(2)}$ must coincide with the position of '$*$' in (A.4b) of the problem $\tilde{E}_N^{(2)}$, and therefore this combination also gives no contribution to $\delta E_N^{(2)}$.

Finally, we are left with the combination of the ground state configurations of the types (A.2d) and (A.4b) which indeed give a finite contribution to $\delta E_N^{(2)}$, equation (A.1). Since the position of one of the elements in the second line of the problem $\tilde{E}_N^{(2)}$ is different from that of the problem $E_N^{(2)}$, in general, the positions of the occupied elements in the rest of the $(N-2)$ lines in these two problems can be quite different. The corresponding energy difference $\delta E_N^{(2)}$, equation (A.1), can be represented as follows:

$$\delta E_N^{(2)} = \overline{\left(\sum_{j=1}^{N}[J_{\tilde{i}(j)j} - J_{i(j)j}]\right)}. \tag{A.6}$$

Here $J_{\tilde{i}(j)j}$ and $J_{i(j)j}$ represent the elements of the jth column occupied in the ground states of the problems $\tilde{E}_N^{(2)}$ and $E_N^{(2)}$, respectively.

Let us consider the integration (the averaging) over the subspace of $J_{i,j}$s such that in the problem $E_N^{(2)}$ the ground state configurations belong to the type (A.2d) *and* in the problem $\tilde{E}_N^{(2)}$ the ground state configurations belong to the type (A.4b) (in the latter case we can always place the occupied position of the second row into the position (2.2)). Since the value of J_{22} in $E_N^{(2)}$ is zero while in $\tilde{E}_N^{(2)}$ this value is non-zero (this is the only difference between the two problems), if we analyse discretely the restrictions imposed on $J_{i,j}$s by the requirements that the ground states are of the type (A.2d) and (A.4b) we can easily see that the corresponding subspace of $J_{i,j}$s in $\tilde{E}_N^{(2)}$ is 'narrower' than that of $E_N^{(2)}$ (in other words, the subspace of $J_{i,j}$s in $E_N^{(2)}$ includes that of $\tilde{E}_N^{(2)}$). Therefore, in the integration over $J_{i,j}$s of the difference $\delta E_N^{(2)}$ the restrictions imposed on $J_{i,j}$s in the two problems do not 'mix': they are defined only by the problem $\tilde{E}_N^{(2)}$.

In the course of integration of the expression (A.6) over $J_{i,j}$s within this subspace we will have all kinds of different ground state configurations in the remaining $(N-2)$ lines $(i = 3, \ldots, N)$ of the two problems. The crucial point, however, is that in terms of the probabilities of these various configurations (within this subspace!) the two problems, $\tilde{E}_N^{(2)}$ and $E_N^{(2)}$, turn out to be symmetric. Let us clarify this point in more detail.

First, let us consider the structure of the sum in equation (A.6) for fixed generic values of the random parameters $J_{i,j}$ such that the ground state of the problem $E_N^{(2)}$ belongs to the type (A.2d), and the ground state of the problem $\tilde{E}_N^{(2)}$ belongs to the type (A.4b). If we combine the occupied positions (denoted by '$\bullet$' and '$*$') in the ground state configurations of the two problems we can find the following generic picture (to simplify the notation it is shown for the matrix with $N = 9$):

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
\otimes & & & & & & & & \\ \hline
 & \bullet & & & & * & & & \\ \hline
 & * & & \bullet & & & & & \\ \hline
 & & * & & & \bullet & & & \\ \hline
 & & \bullet & * & & & & & \\ \hline
 & & & & \bullet & & * & & \\ \hline
 & & & & & & & \bullet & * \\ \hline
 & & & & * & & & & \bullet \\ \hline
 & & & & & & \bullet & * & \\ \hline
\end{array}. \tag{A.7}$$

Here the symbol '$\otimes$' in the position (11) denotes the element (which is equal to zero) occupied simultaneously in both problems. Thus, in the sum (A.6) the first column gives no contribution, then we have two columns (number 2 and 6 in the above example) in which one of the elements of the difference $(J_{\tilde{i}(j)j} - J_{i(j)j})$ belongs to the second line, and finally we have $(N-3)$ *equivalent* columns in which both elements $(\tilde{i}(j), j)$ and $(i(j), j)$ belong to the rest of the $(N-2)$ *equivalent* lines.

By the symmetry of the two problems, $\tilde{E}_N^{(2)}$ and $E_N^{(2)}$, within the subspace of J_{ij}s corresponding to the combination of the classes (A.2d) and (A.4b) we mean the following. Since both problems have *the same* J_{ij}s in $(N-2)$ equivalent lines $(i = 3, \ldots, N)$, and these random parameters have *the same* probability distribution, and in the process of averaging in equation (A.6) we integrate over J_{ij}s within *the same* subspace, the average value of any J_{ij} $(i = 3, \ldots, N;\ j = 3, \ldots, N)$ in the problem $\tilde{E}_N^{(2)}$ must be equal to that in the problem $E_N^{(2)}$. For the same reasons the average value of any J_{i2} $(i = 3, \ldots, N)$ in the problem $\tilde{E}_N^{(2)}$ (provided the element (22) is occupied in the problem $E_N^{(2)}$) must be equal to the average of any J_{ij} $(i = 3, \ldots, N)$ in the problem $E_N^{(2)}$ *provided* the element $(2j)$ is occupied in the problem $\tilde{E}_N^{(2)}$.

For the averaging of the sum in equation (A.6) this has the following consequences. Let us suppose that in a column number j $(j = 3, \ldots, N)$ with some probability P we find the value of the difference $(J_{\tilde{i}(j)j} - J_{i(j)j}) \equiv \delta J$: here the value $J_{\tilde{i}(j)j} \equiv J_1$ is occupied in the problem $\tilde{E}_N^{(2)}$, and the value $J_{i(j)j} \equiv J_2$ is occupied in the problem $E_N^{(2)}$. Then due to the equivalence of the columns $(j = 3, \ldots, N)$ and the lines $(i = 3, \ldots, N)$, and due to the symmetry of the two problems in *another* column j' with *the same probability* P we can find the opposite situation: $(J_{\tilde{i}(j')j'} - J_{i(j')j'}) = -\delta J$. In other words, in another column j' with the same probability P we can find the value $J_2 = J_{\tilde{i}(j')j'}$ occupied in the problem $\tilde{E}_N^{(2)}$ and the value $J_1 = J_{i(j')j'}$ occupied in the problem $E_N^{(2)}$.

A similar situation takes place in the remaining two columns in which one of the elements of the difference $(J_{\tilde{i}(j)j} - J_{i(j)j})$ belongs to the second line. If with some probability P' the value of the occupied element of the second column $J_{\tilde{i}(2)2} = J$ (in the problem $\tilde{E}_N^{(2)}$), then due to equivalence of the lines $(i = 3, \ldots, N)$, and due to the symmetry of the two problems with the same probability P' we can find the same value $J_{i(j)j} = J$ (in the problem $E_N^{(2)}$) of the occupied element in the other column j (number 6 in the example (A.7)) provided the element $(2j)$ of this column is occupied in the problem $\tilde{E}_N^{(2)}$.

Thus, in equation (A.6) we integrate over J_{ij}s the expression which in terms of the elements of the $(N-2)$ lines $(i = 3, \ldots, N)$ is *antisymmetric* with respect to permutations of different equivalent columns. On the other hand, the probability distribution of these elements is *symmetric* with respect to such permutations. Therefore, these elements can be integrated out to give a zero contribution. (The most trivial example of such a situation is an integral of the type $\iint \mathrm{d}J_1\, \mathrm{d}J_2\, (J_1 - J_2) \exp\{-J_1 - J_2\}$: if we integrate here over J_1 and over J_2 in *the same* subspace (whatever it is) this integral is identically equal to zero.) According to the above analysis of the contributions to the energy difference $\delta E_N^{(2)}$ of the other combinations of classes of states (A.2) and (A.4), the integration over the rest of the space of J_{ij}s (out of the subspace corresponding to the considered combination (A.2d) and (A.4b)) gives a zero contribution. In the result, all the parameters J_{ij} $(i = 3, \ldots, N)$ can be integrated out and dropped away from the expression in equation (A.6), and we are left with the averaging of the 'truncated' expression which contains J_{ij}s of the first two lines only:

$$\delta E_N^{(2)} = \boldsymbol{E}\left(\begin{array}{|c||c|c|c|c|c|c|c|} \hline 0 & & & & & & \cdots & \\ \hline 0 & & & & & & \cdots & \\ \hline \end{array}\right) - \boldsymbol{E}\left(\begin{array}{|c|c||c|c|c|c|c|c|} \hline 0 & & & & & & \cdots & \\ \hline & 0 & & & & & \cdots & \\ \hline \end{array}\right). \tag{A.8}$$

It is evident that the ground state energy of the second problem in equation (A.8) is zero, and thus we have proved that the value of $\delta E_N^{(k=2)}$ is given by the rectangular $N \times 2$ problem (2.20).

The generalization of the proof for arbitrary k is straightforward. It can be easily demonstrated on the example of the case $k = 3$:

$$\delta E_N^{(3)} = \boldsymbol{E}\left(\begin{array}{|c|c||c|c|c|c|c|c|}\hline 0 & & & & & & \cdots & \\ \hline & 0 & & & & & \cdots & \\ \hline & 0 & & & & & \cdots & \\ \hline\hline & & & & & & \cdots & \\ \hline & & & & & & \cdots & \\ \hline & & & & & & \cdots & \\ \hline \multicolumn{8}{|c|}{\cdots} \\ \hline & & & & & & \cdots & \\ \hline \end{array}\right) - \boldsymbol{E}\left(\begin{array}{|c|c|c||c|c|c|c|c|}\hline 0 & & & & & & \cdots & \\ \hline & 0 & & & & & \cdots & \\ \hline & & 0 & & & & \cdots & \\ \hline\hline & & & & & & \cdots & \\ \hline & & & & & & \cdots & \\ \hline & & & & & & \cdots & \\ \hline \multicolumn{8}{|c|}{\cdots} \\ \hline & & & & & & \cdots & \\ \hline \end{array}\right)$$

$$\equiv \tilde{E}_N^{(3)} - E_N^{(3)}. \tag{A.9}$$

Here one should make a similar classification to equations (A.2) and (A.4) (which turns out to be only slightly more cumbersome) of all non-equivalent ground state configurations of the problems $E_N^{(3)}$ and $\tilde{E}_N^{(3)}$ according to the positions of the occupied elements of the first three lines. A simple analysis shows that here again the only relevant (for $\delta E_N^{(3)}$) configurations of the problem $E_N^{(3)}$ are those with all three zeros occupied, while in the problem $\tilde{E}_N^{(3)}$ these are the configurations with one or two of the zeros occupied. On the other hand, due to the equivalence of the rest of the $(N-3)$ lines $(i = 4, \ldots, N)$ one finds that all the elements J_{ij} of these lines fall out of the computation. In this way one easily finds that the energy difference $\delta E_N^{(3)}$ is defined only by the elements of the first three lines of the problem $\tilde{E}_N^{(3)}$, which is just defined in terms of the 'truncated' problem:

$$\delta E_N^{(3)} = \boldsymbol{E}\left(\begin{array}{|c|c||c|c|c|c|c|c|}\hline 0 & & & & & & \cdots & \\ \hline & 0 & & & & & \cdots & \\ \hline & 0 & & & & & \cdots & \\ \hline \end{array}\right) \tag{A.10}$$

(the energy of the 'truncated' $N \times 3$ problem $E_N^{(3)}$ is identically equal to zero).

Using the equivalence of the first $(k-1)$ lines in the problem $\tilde{E}_N^{(k)}$ and of the first k lines in the problem $E_N^{(k)}$ a similar procedure can be easily generalized for an arbitrary value of k. Similar to the cases $k = 2$ and 3, here one can easily prove that in the problem $E_N^{(k)}$ the only relevant class of the ground state configurations is that with *all* k zeros (in positions (ii), $i = 1, \ldots, k$) occupied. One can easily see that whenever the ground state of the problem $E_N^{(k)}$ is such that one (or more) of these zeros is not occupied, then the ground state configuration in the problem $\tilde{E}_N^{(k)}$ (defined by the same set of J_{ij}s!) *must* be the same as that of $E_N^{(k)}$, and therefore these types of configurations do not contribute to $\delta E_N^{(k)}$, equations (2.19) and (A.6). Thus, the non-zero contribution to $\delta E_N^{(k)}$ comes only from the subspace of J_{ij}s such that in the ground state of $E_N^{(k)}$ all k diagonal zeros are occupied, while the ground state of $\tilde{E}_N^{(k)}$ (since here one cannot occupy all k zeros) can be any configuration in which one, or two, ... or $(k-1)$ zeros occupied. In other words, in $\tilde{E}_N^{(k)}$ one can have any configuration in which among the first k lines there are $(k-1)$ or $(k-2), \ldots,$ or one line where the occupied position is not the zero one. However complicated these configurations are, in the process of averaging of $\delta E_N^{(k)}$ (of the type (A.6)) over the random parameters J_{ij}, just for symmetry reasons, one again obtains a zero contribution from all J_{ij}s of the last $(N-k)$ lines $(i = k+1, \ldots, N)$, and thus the problem is reduced to the 'truncated' one, equation (2.20), defined by the random parameters J_{ij} of the first k lines only.

The calculation of the actual values of $\delta E_N^{(k)}$ is presented in the next appendix.

Appendix B

In this appendix we prove that

$$\delta E_N^{(k)} = \frac{1}{k(k-1)} \sum_{l=1}^{k-1} \frac{l}{N-l}. \tag{B.1}$$

The solution of the case $k = 2$, equation (A.8), is trivial. Here the ground state configuration is of the type (A.4b), where the position of '$*$' must be at the smallest element out of $2(N-1)$ non-zero J_{ij}s. According to the distribution (1.3), for the average value of this element we obtain

$$\delta E_N^{(2)} = \frac{1}{2(N-1)}. \tag{B.2}$$

Now let us consider a slightly more complicated case $k = 3$, equation (A.10). A simple analysis of the structures of possible ground state configurations shows that all of them can be taken into account in terms of only one 3×3 matrix:

$$\begin{array}{|c|c|c|} \hline 0 & \otimes & z \\ \hline \otimes & 0 & \otimes \\ \hline y & 0 & x \\ \hline \end{array}\,. \tag{B.3}$$

Here x is the smallest element out of $2(N-2)$ equivalent elements J_{2j} $(j = 3, \ldots, N)$ and J_{3j} $(j = 3, \ldots, N)$ of the second and the third lines; y is the smallest element out of two equivalent elements J_{21} and J_{31}; z is the smallest element out of $(N-2)$ equivalent elements J_{1j} $(j = 3, \ldots, N)$ of the first line; the symbol '$\otimes$' denotes the elements which do not enter into any ground state configuration. One can easily check that the matrix in equation (A.10) can have only two ground state energies equal to x or equal to $(y+z)$. Note that if we consider this problem in terms of the 3×3 matrix (B.3), the element x could as well be placed in position $(3, 2)$ (instead of '$\otimes$' which then should be placed at position $(3, 3)$), as well as y could be interchanged with '$\otimes$' in positions $(2, 1)$ and $(3, 1)$.

Now one can easily note that original 3×3 problem (B.3) is actually equivalent to the 2×2 problem

$$\delta E_N^{(3)} = \boldsymbol{E}\left(\begin{array}{|c|c|} \hline 0 & \mathrm{z} \\ \hline \mathrm{y} & \mathrm{x} \\ \hline \end{array}\right) \tag{B.4}$$

where, according to the definitions of the random parameters x, y and z, their statistical distributions are

$$P(x) = 2(N-2)\exp[-2(N-2)x] \tag{B.5}$$

$$P(y) = 2\exp(-2y) \tag{B.6}$$

$$P(z) = (N-2)\exp[-(N-2)z]. \tag{B.7}$$

Keeping in mind further generalization of the solution for arbitrary k, we solve the problem (B.4) in the following way. Similarly to the procedure described at the beginning of section 2, we can 'shift' the elements of the second line (x and y) by the value of the smallest of them, and then integrate it out:

$$\delta E_N^{(3)} = \frac{1}{2(N-1)} + \frac{1}{(N-1)} \boldsymbol{E}\left(\begin{array}{|c|c|} \hline 0 & z \\ \hline 0 & x \\ \hline \end{array}\right). \tag{B.8}$$

The factor $1/(N-1)$ in the second term of the above equation is the probability that y is smaller than x (if the smallest element is x, then the remaining problem will have all zeros at the diagonal, and the minimum energy of this problem is identically equal to zero). The solution of the remaining 2×2 problem is trivial, and eventually we obtain the following result:

$$\delta E_N^{(3)} = \frac{1}{2(N-1)} + \frac{1}{3(N-1)(N-2)} = \frac{1}{3 \times 2}\left[\frac{1}{N-1} + \frac{2}{N-2}\right]. \tag{B.9}$$

Now the generalization of the above procedure for an arbitrary value of k becomes evident. First we note that all possible ground state configurations of the $N \times k$ problem $\delta E_N^{(k)}$, equation (2.20), can be taken into account in terms of the $k \times k$ matrix

$$\begin{array}{|c|c|c|c|c|c|}
\hline
0 & & \ldots & & \otimes & z_1 \\
\hline
 & 0 & \ldots & & \otimes & z_2 \\
\hline
\multicolumn{6}{|c|}{\ldots} \\
\hline
 & & \ldots & 0 & \otimes & z_{(k-2)} \\
\hline
\otimes & \otimes & \ldots & \otimes & 0 & \otimes \\
\hline
y_1 & y_2 & \ldots & y_{(k-2)} & 0 & x \\
\hline
\end{array}. \tag{B.10}$$

Here x is the smallest element out of $2(N-k+1)$ equivalent elements $J_{(k-1)j}$ $(j = k, \ldots, N)$ and J_{kj} $(j = k, \ldots, N)$ of the last two lines; y_j $(j = 1, \ldots, (k-2)$ is the smallest element out of two equivalent elements $J_{(k-1)j}$ and J_{kj}; z_i $(i = 1, \ldots, (k-2)$ is the smallest element out of $(N-k+1)$ equivalent elements J_{ij} $(j = k, \ldots, N)$ of the ith line; and again the symbol '$\otimes$' denotes the elements which do not enter into any ground state configuration.

According to the above definitions of the random parameters $x, \{y_j\}$ and $\{z_i\}$ their probability distribution functions are

$$P(x) = 2(N-k+1)\exp[-2(N-k+1)x] \tag{B.11}$$

$$P(y_j) = 2\exp\left(-2y_j\right) \tag{B.12}$$

$$P(z_i) = (N-k+1)\exp[-(N-k+1)z_i]. \tag{B.13}$$

In this way we can reduce the calculation of $\delta E_N^{(k)}$ to the $(k-1) \times (k-1)$ matrix problem:

$$\delta E_N^{(k)} = \boldsymbol{E}\left(\begin{array}{|c|c|c|c|c|}
\hline
0 & & \ldots & & z_1 \\
\hline
 & 0 & \ldots & & z_2 \\
\hline
\multicolumn{5}{|c|}{\ldots} \\
\hline
 & & \ldots & 0 & z_{(k-2)} \\
\hline
y_1 & y_2 & \ldots & y_{(k-2)} & x \\
\hline
\end{array}\right). \tag{B.14}$$

Taking into account the equivalence of the first $(k-2)$ columns here we can integrate out the smallest element of the last line to obtain

$$\delta E_N^{(k)} = \frac{1}{2(N-1)} + \frac{k-2}{(N-1)}\boldsymbol{E}\left(\begin{array}{|c|c|c|c|c|c|}
\hline
0 & & \ldots & & & z_1 \\
\hline
 & 0 & \ldots & & & z_2 \\
\hline
\multicolumn{6}{|c|}{\ldots} \\
\hline
 & & \ldots & 0 & & z_{(k-3)} \\
\hline
 & & \ldots & & 0 & z_{(k-2)} \\
\hline
y_1 & y_2 & \ldots & y_{(k-3)} & 0 & x \\
\hline
\end{array}\right). \tag{B.15}$$

Now one can easily see that all possible ground state configurations in the remaining $(k-1) \times (k-1)$ problem can be taken into account in the same way as in the previous

$k \times k$ one, equation (B.10). Here we can reduce the number of relevant elements by choosing the smallest one between x and $z_{(k-2)}$, as well as between each y_j of the last line and $J_{(k-2)j}$ $(j = 1, \ldots, (k-3))$ of the previous line. In this way we obtain

$$\delta E_N^{(k)} = \frac{1}{2(N-1)} + \frac{k-2}{(N-1)} \boldsymbol{E} \left(\begin{array}{|c|c|c|c|c|c|} \hline 0 & & \ldots & & \otimes & z_1 \\ \hline & 0 & \ldots & & \otimes & z_2 \\ \hline \multicolumn{6}{|c|}{\ldots} \\ \hline & & \ldots & 0 & \otimes & z_{(k-3)} \\ \hline \otimes & \otimes & \ldots & \otimes & 0 & \otimes \\ \hline y_1 & y_2 & \ldots & y_{(k-3)} & 0 & x \\ \hline \end{array} \right) \tag{B.16}$$

where the random elements x and $\{y_j\}$ according to their definitions are now described by the following distribution functions:

$$P(x) = 3(N-k+1) \exp\left[-3(N-k+1)x\right] \tag{B.17}$$

$$P(y_j) = 3 \exp\left(-3y_j\right) \tag{B.18}$$

while the distribution functions of z_is remain unchanged, equation (B.13). In this way we can reduce the calculation of $\delta E_N^{(k)}$ to the $(k-2) \times (k-2)$ matrix problem

$$\delta E_N^{(k)} = \frac{1}{2(N-1)} + \frac{k-2}{(N-1)} \boldsymbol{E} \left(\begin{array}{|c|c|c|c|c|} \hline 0 & & \ldots & & z_1 \\ \hline & 0 & \ldots & & z_2 \\ \hline \multicolumn{5}{|c|}{\ldots} \\ \hline & & \ldots & 0 & z_{(k-3)} \\ \hline y_1 & y_2 & \ldots & y_{(k-3)} & x \\ \hline \end{array} \right). \tag{B.19}$$

Here again we can integrate out the smallest element of the last line to obtain

$$\begin{aligned} \delta E_N^{(k)} = \frac{1}{2(N-1)} + \frac{k-2}{(N-1)} \Bigg[& \frac{1}{3(N-2)} \\ & + \frac{k-3}{(N-2)} \boldsymbol{E} \left(\begin{array}{|c|c|c|c|c|c|} \hline 0 & & \ldots & & & z_1 \\ \hline & 0 & \ldots & & & z_2 \\ \hline \multicolumn{6}{|c|}{\ldots} \\ \hline & & \ldots & 0 & & z_{(k-4)} \\ \hline & & \ldots & & 0 & z_{(k-3)} \\ \hline y_1 & y_2 & \ldots & y_{(k-4)} & 0 & x \\ \hline \end{array} \right) \Bigg]. \end{aligned} \tag{B.20}$$

Continuing these iterations up to the last trivial 2×2 problem we eventually obtain the following result:

$$\begin{aligned} \delta E_N^{(k)} = \frac{1}{2(N-1)} + \frac{k-2}{(N-1)} \Bigg[& \frac{1}{3(N-2)} + \frac{k-3}{(N-2)} \Bigg[\frac{1}{4(N-3)} \\ & + \frac{k-4}{(N-3)} \Bigg[\cdots \Bigg[\frac{1}{k(N-k+1)} \Bigg] \cdots \Bigg] \Bigg] \Bigg]. \end{aligned} \tag{B.21}$$

After simple algebra the above expression can be easily reduced to the following form:

$$\delta E_N^{(k)} = \frac{1}{k(k-1)} \left[\frac{1}{N-1} + \frac{2}{N-2} + \cdots + \frac{k-1}{N-k+1} \right] \tag{B.22}$$

which proves equation (B.1).

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